

IPP-QM-1: Basic quantum formalism

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MT25

The course

1. Basic quantum formalism
2. Density operators and entanglement
3. Decoherence
4. The measurement problem
5. Dynamical collapse theories
6. Bohmian mechanics
7. Everettian structure
8. Everettian probability
9. EPR and Bell's theorem
10. The Bell-CHSH inequalities and possible responses
11. Contextuality
12. The PBR theorem
13. Quantum logic
14. QBism
15. Pragmatism and relational quantum mechanics
16. Wavefunction realism

Books etc.

- ▶ David Albert, *Quantum Mechanics and Experience*, Boston: Harvard University Press, 1994.
- ▶ Tim Maudlin, *Philosophy of Physics Volume II: Quantum Mechanics*, Princeton: Princeton University Press, 2019.
- ▶ Tim Maudlin, *Quantum Non-Locality and Relativity*, third edition, Oxford: Wiley-Blackwell, 2011.
- ▶ David Wallace, *The Emergent Multiverse*, Oxford: Oxford University Press, 2012.

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- ▶ John S. Bell, “Against Measurement”, in *Speakable and Unspeakable in Quantum Mechanics*, second edition, Cambridge: Cambridge University Press, 2004.
- ▶ David Wallace, “Philosophy of Quantum Mechanics”, in D. Rickles (ed.), *The Ashgate Companion to Contemporary Philosophy of Physics*, London: Routledge, 2008.

Formalism

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Highly recommended are:

- ▶ David Wallace, “The Formalism of Quantum Mechanics”, 2005.
- ▶ David Wallace, “The Formalism of Quantum Mechanics II – Density Operators and Entanglement”, 2005.
- ▶ Guido Bacciagaluppi, “Density Operators in Quantum Mechanics”, 1998.
- ▶ Frederic P. Schuller, “Lectures on Quantum Theory”, 2015.

(All on Canvas.)

‘Physics-first’ versus ‘math-first’ approaches

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- ▶ In mathematics textbooks, one might find that mathematics/geometry introduced from the outset, and then used to *explain* the relevant physics. (This would be a ‘math-first’ approach.)
- ▶ This should remind you of dynamics-vs-geometry debates in IPP-SR! But I won’t dwell on it further here.

Today

Quantum states

Hilbert spaces

More about Hilbert spaces

Linear operators

The physical significance of operators

Wavefunctions and infinite-dimensional Hilbert spaces

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 1. if the system is a one-dimensional spinless particle then *position* is maximally specific;
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- ▶ Write the states of definite m as $|m_1\rangle, |m_2\rangle, \dots, |m_n\rangle$.
- ▶ $|m_i\rangle$ is a state such that, if we measure M when the system is in that state (assuming that we have an ideal measuring device), then we definitely get the result m_i .

Superpositions

- ▶ In classical physics, this would be the whole story: quantities always have measurable values, so the system would be in one of the $|m_i\rangle$, and we just ask which.

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$$|\psi\rangle = \alpha_1 |m_1\rangle + \alpha_2 |m_2\rangle + \dots + \alpha_n |m_N\rangle, \quad \forall \alpha_j \in \mathbb{C}.$$

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- ▶ Because probabilities sum to one, this implies that quantum states must be *normalised*,

$$\sum_{i=1}^N |\alpha_i|^2 = 1.$$

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- ▶ Elements of a Hilbert space needn't be normalised (only the physical ones are), and are called *vectors*.

Hilbert space inner product

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- ▶ If $|\psi\rangle = \sum_i \alpha_i |m_i\rangle$ and $|\phi\rangle = \sum_i \beta_i |m_i\rangle$ are vectors, then their inner product is defined as

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 1. *Linear*: $\langle\phi|(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha\langle\phi|\psi_1\rangle + \beta\langle\phi|\psi_2\rangle$
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 3. *Positive*: If $|\psi\rangle \neq 0$ then $\langle\psi|\psi\rangle \in \mathbb{R}^+$.
- ▶ We can write the normalisation condition on states as:
 $|\psi\rangle$ is normalised iff $\langle\psi|\psi\rangle = 1$.

Mathematical approaches to Hilbert space

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- ▶ See e.g. the Schuller notes for more details.

Bases for Hilbert space

- ▶ A *basis* for a Hilbert space \mathcal{H} is a set of vectors in \mathcal{H} s.t.:
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- ▶ It is typical to insist that a basis be *orthonormal*, which means that it also satisfies:
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- ▶ **Exercise:** Prove that any two bases for \mathcal{H} have the same number of vectors.
- ▶ Let $|\psi\rangle \in \mathcal{H}$ and let $\{|i\rangle\}$ be any orthonormal basis. By definition, there will be complex numbers $\{\alpha_i\}$ s.t. $|\psi\rangle = \sum_i \alpha_i |i\rangle$. Because the basis is orthonormal, $\langle i|\psi\rangle = \alpha_i$; that is, we have the useful expansion

$$|\psi\rangle = \sum_i |i\rangle \langle i|\psi\rangle$$

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- ▶ If $|\psi\rangle$ is an arbitrary state, what's the probability of getting result k_i when we measure K ?
- ▶ It's a *postulate* of 'standard', textbook quantum mechanics that the probability is $|\langle k_i|\psi\rangle|^2$.

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- ▶ States of that system are represented by normalised vectors in that Hilbert space.
- ▶ A maximally specific measurement is represented by a basis of the Hilbert space.
- ▶ If K is a maximally specific measurement and $|k_i\rangle$ has definite value k_i of K , then the probability is of getting k_i when we measure K on the state $|\psi\rangle$ is $|\langle k_i|\psi\rangle|^2$.

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- ▶ It's a smaller Hilbert space, entirely contained within the original one.
- ▶ Such spaces are called *subspaces*.

Global phase transformations

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- ▶ So strictly, it isn't states in Hilbert space which represent physical systems—it's equivalence classes of states related by global phase, sometimes called *rays*.
- ▶ Practically, however, it's easier to just carry on working with vectors in Hilbert space.

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- ▶ For defences of the mainstream, see (Wallace 2022) or (Gao 2024); I won't question it further in these lectures.

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- A *linear operator* \hat{A} is a map from Hilbert space to itself (i.e., something which eats states and spits out states) which has the linearity property:

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- ▶ Simple example: $|\phi\rangle \langle\chi|$, where $|\psi\rangle$ and $|\chi\rangle$ are arbitrary (normalised) vectors; this is defined as follows:
 $(|\phi\rangle \langle\chi|)|\psi\rangle = |\phi\rangle \times \langle\chi|\psi\rangle.$

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- ▶ E.g., acting on $|\psi\rangle = \alpha|\phi\rangle + \beta|\chi\rangle$ with $|\phi\rangle\langle\phi|$ yields $\alpha|\phi\rangle$ (where $|\phi\rangle$ and $|\chi\rangle$ are part of an orthonormal basis for the Hilbert space).

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- ▶ E.g., acting on $|\psi\rangle = \alpha|\phi\rangle + \beta|\chi\rangle$ with $|\phi\rangle\langle\phi|$ yields $\alpha|\phi\rangle$ (where $|\phi\rangle$ and $|\chi\rangle$ are part of an orthonormal basis for the Hilbert space).
- ▶ More general (i.e., higher-dimensional) projectors don't just project out one particular component of a vector: they project out all components in some particular subspace.

Resolutions of the identity

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- ▶ The limiting case of a projector is one which projects onto the entire Hilbert space—and so leaves the vector alone!
- ▶ The *identity operator* can be written in the same way as any other projector:

$$\hat{1} = \sum_i |n_i\rangle \langle n_i|,$$

where $\{|n_i\rangle\}$ is any basis. An expression like this for $\hat{1}$ is called a *resolution of the identity*.

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- ▶ This means that we can insert resolutions of the identity into quantum expressions wherever we please, e.g.:
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 - ▶ $\hat{A} = \hat{1}\hat{A}\hat{1} = (\sum_i |n_i\rangle \langle n_i|) \hat{A} (\sum_j |n_j\rangle \langle n_j|) = \sum_{i,j} |n_i\rangle \langle n_j| \langle n_i| \hat{A} |n_j\rangle$.

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- ▶ This gives us the components A_{ij} of the matrix associated with the operator \hat{A} in the $\{|n_i\rangle\}$ basis.

Eigenvalues and eigenvectors

- Recall that an *eigenvector* \mathbf{v} of a matrix \mathbf{M} is a column vector satisfying $\mathbf{M}\mathbf{v} = \lambda\mathbf{v}$, and the number λ is called the *eigenvalue* associated with \mathbf{v} .

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- ▶ (So, non-degenerate operators have only one-dimensional eigensubspaces.)

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- ▶ (For further details on the spectral theorem, see e.g. the Schuller notes.)
- ▶ If \hat{C} is non-degenerate, then there's a *unique* such basis.

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- ▶ If \hat{A} and \hat{B} are normal operators which commute, then there exists a normal operator \hat{C} and functions f, g such that $\hat{A} = f(\hat{C})$ and $\hat{B} = g(\hat{C})$.

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- ▶ Conversely, the spectral theorem means that every non-degenerate self-adjoint operator determines a maximally specific measurement.

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- ▶ However, for a recent and philosophically rich article arguing that other normal operators could serve just as well (despite having complex eigenvalues!), see Roberts (2017).

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 - ▶ For example, a photon detector destroys a photon completely when measuring its location!

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- ▶ So if we measure \hat{C} then we automatically measure \hat{A} and \hat{B} , just by applying the appropriate functions to the outcome.
- ▶ We conclude that *commuting operators can be measured simultaneously*.
- ▶ Conversely, if two operators don't commute then it's at best unclear what it would mean to measure them simultaneously.

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- ▶ Similarly, *unitary operators* can be defined either as (i) operators satisfying $\hat{A}^\dagger \hat{A} = \hat{1}$, or (ii) operators whose eigenvalues all have modulus 1 (so of the form $e^{i\theta}$).
- ▶ The significance of unitary operators is that they map physical states to physical states, for if a map is to do this then it must preserve normalisation: if $|\psi\rangle$ is a physical state, then $|\psi'\rangle = \hat{U}|\psi\rangle$ is a physical state only if

$$1 = \langle\psi'|\psi'\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle,$$

which is true for all states only if $\hat{U}^\dagger\hat{U} = \hat{1}$.

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1. *Time evolution*: It's a postulate of textbook quantum mechanics that states evolve linearly: so if $|\psi\rangle$ evolves into $|\psi'\rangle$ and $|\phi\rangle$ evolves into $|\phi'\rangle$, then $\alpha|\psi\rangle + \beta|\phi\rangle$ evolves into $\alpha|\psi'\rangle + \beta|\phi'\rangle$. This means that time evolution is described by unitary operators: for any times t, t_0 , there must be a unitary operator $\hat{U}(t, t_0)$ which evolves states at time t_0 into states at time t :

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2. *Symmetry transformations*: Symmetries of quantum systems, such as rotations or translations, are described by (anti-)unitary operators. (See e.g. Weinberg 1995)

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- ▶ Then, acting with this operator on some state $|\psi(t)\rangle$ yields

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle.$$

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- ▶ (For more, see the seminal von Neumann (1932).)

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- ▶ The *wavefunction* is defined as $\psi(x) := \langle x|\psi\rangle$.

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- ▶ After all, it's a function of three coordinates, labelled 'positions', and when it is concentrated in some region, the probability is very high that a measurement of position will give a result in that region.
- ▶ But this is a mistake! Consider the wavefunction for *two* particles: this must be a (complex-valued) function of *six* variables: three for the position of the first particle, and three for the position of the second.

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- ▶ But this is a mistake! Consider the wavefunction for *two* particles: this must be a (complex-valued) function of *six* variables: three for the position of the first particle, and three for the position of the second.
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- ▶ So, the wavefunction can't straightforwardly be thought of as living on space at all.
- ▶ Rather, the wavefunction is a complex-valued field on *configuration space*, which is the space of all possible coordinates of all particles, with $3N$ dimensions for an N -particle system.






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- ▶ Tomorrow, we'll look at *density operators* and *entanglement*.

References

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